Fourier Decomposition in Hilbert Spaces and Wavelet Sets

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Our project will provide a rigorous introduction to Hilbert spaces using basic tools of real analysis. We will then apply this theory to the problem of Fourier series decomposition in arbitrary Hilbert spaces. The second part of our project will extend the notion of Fourier series into Fourier transforms and then use this idea to develop and apply the theory of Wavelets. Applications will include graphs of various wavelets in MAPLE.

Fourier Series Approximation

Theorem 1 *If $f(x)$ is piecewise continuous on $[-\pi, \pi]$ and standardized, then*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n cos(nx) + b_n sin(nx)] \, \forall x \in [-\pi, \pi]$$

Where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) cos(nx) dx b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) sin(nx) dx$$

Fourier Decomposition in Arbitrary Hilbert Spaces

Theorem 2 *Let X be any arbitrary Hilbert space.*
Now, $\forall S = \{s_\alpha : \alpha \in I\}$, an orthonormal basis in X.

$$x = \sum_{\alpha \in I} < x, s_\alpha > s_\alpha \ \ \forall x \in X$$

Definition 1 (Fourier Coefficients) *In an arbitrary Hilbert space, the numbers $< x, s_\alpha >$ are called the Fourier coefficients of x relative to S.*

Definition 2 (Fourier Series) *In an arbitrary Hilbert space, the series $\sum_{\alpha \in I} < x, s_\alpha > s_\alpha$ is called the Fourier Series of x relative to S.*

Fourier Transform

A Fourier series is defined on the interval $[-\pi, \pi]$. This interval works nicely, but now we will investigate what happens when we let this interval be $[-\ell, \ell]$ and let $\ell \to \infty$. When we allow this interval, we get the *Fourier Transform*. Which will be useful in the study of wavelets.

Theorem 3 *If f is a continuously differentiable function with* $\int_{-\infty}^{\infty} |f(t)|dt < \infty$, *then*

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(\lambda) e^{\imath\lambda x} d\lambda$$

where $\widehat{f}(\lambda)$ *(The Fourier Transform of F) is given by*

$$\widehat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-\imath\lambda t} dt$$

Orthogonal Wavelets and Wavelet Sets

Now we will introduce the notion of a wavelet. This will involve the theory of the Fourier Transform and some characterization theorems of wavelets which prove they can be translated and dilated to cover $\mathbb{R}$. Once wavelets have been introduced theoretically we will consider some examples to illustrate the various properties of wavelets.

Theory of Wavelets

We can say a function is in the Hilbert Space, $f \in L^2(0, 2\pi)$ if $\int_{-\pi}^{\pi} \mid f(x) \mid^2 dx < \infty$. So we see that it can be generated by a basis of the form $e^{\imath x}$. Since these are actually of the form $cosx + \imath sinx$, we can think of these functions as waves. However the space we need to consider is $L^2(\mathbb{R})$. If we recall the inner product

$$< f, g >= \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx$$

we see that these functions must satisfy the following:

$$\int_{-\infty}^{\infty} \mid f(x) \mid^2 dx < \infty$$

This means that sin and cos will no longer work as a basis for this set. In fact any periodic function will fail because functions in $L^2(\mathbb{R})$ need to decay to zero at infinity. Therefore we cannot properly call our basis elements waves, rather they are "little waves" or wavelets.

Theory of Wavelets

Definition 3 (Orthogonal Dyadic Wavelet) *An Orthogonal Dyadic Wavelet is any $L^2(\mathbb{R})$ function ψ such that $S = \{2^{\frac{n}{2}}\psi\,(2^n t - \ell) : n, \ell \in \mathbb{Z}\}$ constitutes an orthogonal basis for $L^2(\mathbb{R})$.*

Definition 4 (Wavelet set) *A measurable subset $E \subset \mathbb{R}$ is called a wavelet set if $\frac{1}{\sqrt{2\pi}}\chi_E$ is the Fourier transform of a wavelet.*

Definition 5 (Translation congruent modulo 2π) *We say that measurable sets E, F are translation congruent modulo 2π if there is a measurable bijection $\psi : E \to F$ such that $\psi(s) - s$ is an integral multiple of 2π for each $s \in E$.*

Definition 6 (Dilation congruent modulo 2) *We say that measurable sets G, H are dilation congruent modulo 2 if there is a measurable bijection $\tau : G \to H$ such that for each $s \in G$ there is an integer n, depending on s, such that $\tau(s) = 2^n s$.*

Theorem 4 *Let $E \subset \mathbb{R}$. E is a wavelet set if and only if E is both translation congruent to* $[0, 2\pi) \bmod 2\pi$ *and dilation congruent to* $[-2\pi, -\pi) \cup [\pi, 2\pi) \bmod 2$.